Quantum particles trapped in a position-dependent mass barrier; a d-dimensional recipe

Omar Mustafa[1] and S. Habib Mazharimousavi[2]
Department of Physics, Eastern Mediterranean University,
G Magusa, North Cyprus, Mersin 10,Turkey
[1]e-mail: omar.mustafa@emu.edu.tr
[2]e-mail: habib.mazhari@emu.edu.tr

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Abstract

We consider a free particle, $V(r) = 0$, with a position-dependent mass $m(r) = 1/(1+\varsigma^2 r^2)^2$ in the d-dimensional Schrödinger equation. The effective potential turns out to be a generalized Pöschl-Teller potential that admits exact solution.

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1 Introduction

Quantum mechanical particles endowed with position - dependent mass (PDM), $M(r)$, have attracted attention and inspired intense research activities over the years [1-16] They form interesting and useful models for many physical problems. They are used, for example, in the energy density many-body problem [2], in the determination of the electronic properties of the semiconductors [3] and quantum dots [4], in quantum liquids [5], in the ^3He clusters [6] and metal clusters [7], in the Bohmian approach to quantum theory [8], full/partial Gaussian wave-packet revival inside an infinite potential [9], etc.

In addition to its practical applicability side, conceptual problems of delicate nature erupt in the study of quantum mechanical systems with position-dependent mass (e.g., momentum operator does not commute with $m(r)$, uniqueness of kinetic energy operator, etc.). Comprehensive discussion on such issues could be found in, e.g., [8,10, and related references therein].

On the other hand, the exact solvability of the d-dimensional Schrödinger equation may very well form the major ingredient for methods that are based on a Liouvillean-type change of variables (cf., e.g., [17-19]) such as the point canonical transformation (PCT) method (cf., e.g., [13,14]). Where, in such

methodologies, the exact solution of the so-called *reference Schrödinger equation* (eigenvalues and eigenfunctions) is mapped into an exact solution (eigenvalues and eigenfunctions) of the so-called *target Schrödinger equation* (cf, e.g. [1,14]).

In a recent study, we have introduced a d-dimensional regularization of the PCT-method for some PDM-quantum mechanical particles [1]. Therein, inter-dimensional degeneracies associated with the isomorphism between the angular momentum quantum number ℓ and dimensionality d are incorporated through the central repulsive/attractive core $\ell\left(\ell+1\right)/r^{2}\longrightarrow\ell_{d}\left(\ell_{d}+1\right)/r^{2}$ (where $\ell_{d}=\ell+\left(d-3\right)/2$ for $d\geq2$) of the spherically symmetric Schrödinger equation (cf, e.g., [20], Gang in [10] and Quesne in [10] for more details).

In this work, we study "*free*" particles, $V\left(r\right)=0$, trapped in their own position-dependent mass barriers (hence, labeled as *quasi-free* particles), where inter-dimensional degeneracies remain intact in the *d-dimensional radial Schrödinger equation* (in atomic units $\hbar=m_{\circ}=1$, the position-dependent mass $M\left(r\right)=m_{\circ}\,m\left(r\right)$, and with $\alpha=\gamma=0$ and $\beta=-1$ in Eq.(1.1) of Tanaka in [10].)

$$\left\{\frac{d^{2}}{dr^{2}}-\frac{\ell_{d}\left(\ell_{d}+1\right)}{r^{2}}+\frac{m'\left(r\right)}{m\left(r\right)}\left(\frac{d-1}{2r}-\frac{d}{dr}\right)+2m\left(r\right)E\right\}R_{n_{r},\ell}\left(r\right)=0. \quad (1)$$

Where $n_{r}=0,1,2,\cdots$ is the radial quantum number, and $m'\left(r\right)=dm\left(r\right)/dr$. Moreover, the $d=1$ can be obtained through $\ell_{d}=-1$ and $\ell_{d}=0$ for even and odd parity, $\mathcal{P}=\left(-1\right)^{\ell_{d}+1}$, respectively (cf, e.g., Mustafa and Znojil in [20]). Nevertheless, the inter-dimensional degeneracies associated with the isomorphism between angular momentum ℓ and dimensionality d builds up the ladder of excited states for any given n_{r} and nonzero ℓ from the $\ell=0$ result, with that n_{r}, by the transcription $d\rightarrow d+2\ell$. That is, if $E_{n_{r},\ell}\left(d\right)$ is the eigenvalue in d-dimensions then

$$E_{n_{r},\ell}\left(2\right)\equiv E_{n_{r},\ell-1}\left(4\right)\equiv\cdots\equiv E_{n_{r},1}\left(2\ell\right)\equiv E_{n_{r},0}\left(2\ell+2\right) \quad (2)$$

for even d, and

$$E_{n_{r},\ell}\left(3\right)\equiv E_{n_{r},\ell-1}\left(5\right)\equiv\cdots\equiv E_{n_{r},1}\left(2\ell+1\right)\equiv E_{n_{r},0}\left(2\ell+3\right) \quad (3)$$

for odd d. For more details on inter-dimensional degeneracies the reader may refer to a sample of references in [20].

With the PCT-method in point (cf, e.g., [1]), a substitution of the form $R\left(r\right)=g\left(r\right)\,\phi\left(q\left(r\right)\right)$ in (1) would result in $g\left(r\right)^{2}q'\left(r\right)=m\left(r\right)$, manifested by the requirement of a vanishing coefficient of the first-order derivative of $\phi\left(q\left(r\right)\right)$ (hence a one-dimensional form of Schrödinger equation is achieved), and $q'\left(r\right)^{2}=m\left(r\right)$ to avoid position-dependent energies-multiplicity (i.e., $2E\,m\left(r\right)/q'\left(r\right)^{2}\Longrightarrow 2E$). Hence,

$$q\left(r\right)=\int^{r}\sqrt{m\left(t\right)}dt\;\;\Longrightarrow\;g\left(r\right)=m\left(r\right)^{1/4}. \quad (4)$$

This in effect implies

$$\left\{-\frac{1}{2}\frac{d^{2}}{dq^{2}}+V_{eff}\left(q\left(r\right)\right)\right\}\phi_{n_{r},\ell_{d}}\left(q\left(r\right)\right)=E_{d}\phi_{n_{r},\ell_{d}}\left(q\left(r\right)\right), \quad (5)$$

with an effective potentials

$$V_{eff}\left(q\left(r\right)\right)=\frac{\ell_d\left(\ell_d+1\right)}{2r^2m\left(r\right)}-U_d\left(r\right) \tag{6}$$

where

$$U_d\left(r\right)=\frac{m''\left(r\right)}{8m\left(r\right)^2}-\frac{7m'\left(r\right)^2}{32m\left(r\right)^3}+\frac{m'\left(r\right)\left(d-1\right)}{4r\,m\left(r\right)^2}. \tag{7}$$

2 Consequences of an asymptotically vanishing mass settings as $r\longrightarrow\infty$

A "*free*" particle with an asymptotically vanishing position-dependent mass $m(r)=1/(1+\varsigma^2r^2)^2$ would experience an effective potential

$$V_{eff}\left(q\left(r\right)\right)=\frac{\varsigma^2}{2}\left[\frac{\varkappa(\varkappa-1)}{\sin^2(\varsigma q)}+\frac{\lambda(\lambda-1)}{\cos^2(\varsigma q)}\right]-\frac{\varsigma^2}{2}, \tag{8}$$

where

$$q(r)=\frac{1}{\varsigma}\arctan(\varsigma r)\Rightarrow\varsigma r=\tan(\varsigma q), \tag{9}$$

and

$$U_d(r)=-\left(\varsigma^2d\right)\tan^2\left(\varsigma q\right)+\frac{\varsigma^2}{2}(1-2d) \tag{10}$$

In such settings, Eq. (5) reads

$$\left\{-\frac{1}{2}\frac{d^2}{dq^2}+\frac{\varsigma^2}{2}\left[\frac{\varkappa(\varkappa-1)}{\sin^2(\varsigma q)}+\frac{\lambda(\lambda-1)}{\cos^2(\varsigma q)}\right]\right\}\phi_{n_r,\ell_d}\left(q\right)=\varepsilon\phi_{n_r,\ell_d}\left(q\right), \tag{11}$$

where

$$\varkappa(\varkappa-1)=l_d(l_d+1),\ \ \lambda(\lambda-1)=l_d(l_d+1)+2d\quad\text{and }\varepsilon=E+\frac{1}{2}\varsigma^2. \tag{12}$$

Equation (11) is obviously a standard one-dimensional form of Schrödinger equation with a generalized Pöschl-Teller effective potential which admits exact solution of the form

$$\epsilon_{n_r}=\frac{\varsigma^2}{2}(\varkappa+\lambda+2n_r)^2 \tag{13}$$

$$\phi_{n_r,\ell_d}\left(q\right)=C\sin^{\varkappa}(\varsigma q)\cos^{\lambda}(\varsigma q)\ {}_2F_1(-n_r,\varkappa+\lambda+n_r,\varkappa+\frac{1}{2};\sin^2(\varsigma q)) \tag{14}$$

with $\varkappa,\lambda>1$, $\phi_{n_r,\ell_d}\left(0\right)=0$ and $\phi_{n_r,\ell_d}\left(\frac{\pi}{2\varsigma}\right)=0$, as reported by Salem and Montemayor (see Eq.(4.7) in [21]). This in turn would lead to

$$E_{n_r,l_d}=\frac{\varsigma^2}{2}((c+\frac{1}{2}\Delta+2n_r)^2-1);\ \ \Delta=\sqrt{(2l_d+1)^2+8d} \tag{15}$$

$$R_{n_r,l_d}(r) = \tilde{C}\,\rho^{l_d+1}(1+\rho^2)^{-\frac{1}{4}(2l_d+5+\Delta)}\ {}_2F_1(-n_r, c+\frac{\Delta}{2}+n_r, c; \frac{\rho^2}{1+\rho^2}) \quad (16)$$

where $\rho = \varsigma r$, and $c = l_d + \frac{3}{2}$.

However, for $\varkappa = 0, 1$ (a requirement suggested by relation (12) when $\ell_d = 0, -1$) the effective potential in (11) collapses into

$$V_{eff}(q\,(r)) = \frac{\varsigma^2}{2}\frac{\lambda(\lambda-1)}{\cos^2(\varsigma q)}. \quad (17)$$

Which admits an exact solution

$$E_{n_r} = 2\varsigma^2(n_r + \frac{\lambda}{2})^2 - \frac{\varsigma^2}{2} \quad (18)$$

$$\phi_{n_r,\ell_d}(q) = A\ \cos^{\lambda}(\varsigma q)\ {}_2F_1(-n_r, n_r+\lambda, \frac{1}{2}, \sin^2(\varsigma q)), \quad (19)$$

and consequently

$$E_{n_r,0} = 2\varsigma^2(n_r + \frac{\lambda}{2})^2 - \frac{\varsigma^2}{2} \quad (20)$$

$$R_{n_r}(r) = \tilde{A}\ (1+\rho^2)^{-\frac{1}{4}(2l_d+5+\Delta)}\ {}_2F_1(-n_r, c+\frac{\Delta}{2}+n_r, c; \frac{\rho^2}{1+\rho^2}). \quad (21)$$

where $\lambda = (1+\Delta)/2$.

3 Concluding Remarks

In this letter, we considered a quasi-free particle with an asymptotically vanishing position-dependent mass $m(r) = 1/(1+\varsigma^2 r^2)^2$ and radial potential $V(r) = 0$ (i.e., "free" particle in this sense). We have shown that under these settings the particle experiences an effective potential of the form of a Pöschl-Teller, Eq.(8). The exact solution of which is mapped to match the attendant settings of our quasi-free particle with the above mentioned position-dependent mass.

References

[1] O Mustafa and S.H Mazharimousavi 2006 ”Point canonical transformation d-dimensional regularization” (arXiv: math-ph/0602044)

[2] A Puente and M Casas Comput. Mater Sci. **2** (1994) 441

[3] Bastard G 1988 *”Wave Mechanics Applied to Semiconductor Heterostructures” ,* Les Editions de Physique, Les Ulis

[4] L I Serra and E Lipparini Europhys. Lett. **40** (1997) 667

[5] F Arias de Saaverda, J Boronat, A Polls , and A Fabrocini Phys. Rev. **B 50** (1994) 4248

[6] M Barranco , M Pi , S.M Gatica ., E.S Hemandez ., and J. Navarro Phys. Rev. **B 56** (1997) 8997

[7] A Puente , L I Serra, and M Casas Z. Phys. **D 31** (1994) 283

[8] A R Plastino, M Casas and A. Plastino Phys. Lett. **A281(**2001) 297 (and related references therein)

[9] A Schmidt Phys. Lett. **A** (2006) (in press)

[10] T Tanaka J. Phys. **A 39** (2006) 219

C Quesne, Ann. Phys. **321** (2006) 1221

C Gang, Phys. Lett. A **329** (2004) 22

A R Plastino, A Rigo, M Casas, F Garcias, and A. Plastino Phys. Rev. **A 60** (1999) 4318

[11] S H Dong and M. Lozada-Cassou Phys. Lett. **A 337** (2005) 313

I O Vakarchuk J. Phys. **A**; Math and Gen **38** (2005) 4727

C Y Cai, Z Z Ren and G X Ju Commun. Theor. Phys. **43** (2005)1019

[12] B Bagchi, A Banerjee, C Quesne and V M Tkachuk J. Phys. **A**; Math and Gen **38** (2005) 2929

J Yu and S H Dong Phys. Lett. **A 325** (2004)194

L Dekar, L Chetouani and T F Hammann J. Math. Phys. **39** (1998) 2551

[13] C Quesne and V M Tkachuk J. Phys. **A**; Math and Gen **37** (2004) 4267

L Jiang, L Z Yi, and C S Jia Phys. Lett. **A 345** (2005) 279

A D Alhaidari Int. J. Theor. Phys. **42** (2003) 2999

[14] A D Alhaidari Phys. Rev. **A 66** (2002) 042116

[15] R De, R Dutt and U Sukhatme J. Phys. **A**; Math and Gen **25** (1992) L843

[16] G Junker J. Phys. **A**; Math and Gen **23** (1990) L881

[17] J Liouville J. Math. Pure Appl. **1** (1837)16

[18] M Znojil and G Lévai J. Math. Phys. **42** (2001)1996

M Znojil "$\mathcal{PT}$-symmetric form of the Hulthén potential" (2000) (arXiv: math-ph/0002017)

[19] M Znojil and G Lévai Phys. Lett. **A 271** (2000) 327

[20] D R Herschbach et al *"Dimensional Scaling in Chemical Physics"* (Kluwer Academic Publishers 1993, Dordrecht, Netherlands.)

D R Herschbach J. Chem. Phys. **84** (1986) 838

H Taseli J. Math. Chem. **20** (1996) 235

O Mustafa and M Znojil J. Phys.A; Math and Gen. **35** (2002) 8929

O Mustafa and M Odeh J. Phys. A; Math and Gen. **32** (1999) 6653

O Mustafa and M Odeh J. Phys. A; Math and Gen. **33** (2000) 5207

M M Nieto Am. J. Phys. **47** (1979) 1067

[21] L D Salem and R Montemayor Phys. Rev. **A 47** (1993) 105

S Flügge 1974, *Practical Quantum Mechanics,* Springer, Berlin.